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Abstract

The purpose of this article is two-fold. In the first place, we prove that a set is the image of a non empty closed convex subset of a real Banach space under an onto Fredholm operator of positive index if and only if it can be written as the union of $\{D_n : n \in \mathbb{N}\}$, a non-decreasing family of non empty, closed, convex and bounded sets such that $D_n + D_{n+2} \subseteq 2\, D_{n+1}$ for every $n \in \mathbb{N}$.

The second part of this article proves that in every infinite dimensional real Banach space there is a convex set which can be expressed as the union of countably many closed sets, but not as the union of countably many closed and convex sets. Accordingly, every infinite dimensional real Banach space contains a convex F_σ set which is not the image of a closed convex set under a Fredholm operator.

Keywords: Fredholm operator, image of a closed convex set, countable union of closed convex sets, F_σ set
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1. Introduction

This article deals with one of the most ubiquitous concepts in convex analysis and optimization: the image of a closed and convex set under a Fredholm operator. The case of an operator acting between two finite dimensional Banach spaces is well understood. With virtually no modifications, the technique used by Klee to prove [6, Theorem 6.1] (a different method may

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be retrieved from the proof given by Sang and Tam to [8, Proposition 2.2]), provides us with the following result.

Theorem [Klee]: *Let X and Y be euclidean spaces, and $\Pi : Y \to X$ be an onto Fredholm operator of positive index. Then the set C is the image of a non empty closed and convex subset of Y under Π if and only if it is a non empty convex F_σ subset of X.*

Our first result (Theorem 2.3, Section 2) is a generalization of the previous Theorem to the setting of possibly infinite dimensional real Banach spaces.

Theorem A: *Let X and Y be real Banach spaces, and $\Pi : Y \to X$ be an onto Fredholm operator of positive index. Then the set C is the image of a non empty closed and convex subset of Y under Π if and only if it is the union of $\{D_n : \ n \in \mathbb{N}\}$, a non-decreasing family of non empty, closed, convex and bounded sets, with the additional property that*

$$D_n + D_{n+2} \subseteq \ 2\, D_{n+1} \qquad \forall n \in \mathbb{N}. \tag{1}$$

The rather technical assumption of Theorem A may be justified by arguing that, on one hand, any non empty closed and convex subset of a Banach space may easily be expressed as the union a a non-decreasing family composed of countably many non empty bounded closed and convex sets fulfilling relation (1), and on the other, that the image under a Fredholm operator of such a family is again a nondecreasing sequence of bounded closed and convex sets satisfying relation (1). However, when X is a finite dimensional Banach space, by combining the two above-mentioned theorems, we deduce that a set is the union of a non-decreasing family of non empty, closed, convex and bounded sets fulfilling relation (1) if and only if it is a non empty convex F_σ set.

Our second result (Theorem 4.3, Section 4) addresses the interplay between the hypothesis of Theorem A, on one hand, and of the previously mentioned Theorem by Klee, on the other, in the case of an infinite dimensional real Banach space.

Theorem B: *Every infinite dimensional real Banach space contains a convex set which can be expressed as the union of countably many closed sets, but not as the union of countable many closed and convex sets.*

In other words, the appropriate notion to describe the image of a closed convex set under a Fredholm operator is indeed the union of an increasing family of non empty closed, convex and bounded sets satisfying the property (1), as stated by Theorem A.

As a first step in achieving Theorem B, we are concerned with the convex hull of the extreme points of the unit ball in ℓ_∞. Indeed, as stated by Albiac and Kalton at page 29 of their textbook, [1], "[the] structure [of ℓ_∞] has been well understood for many years. However, there can still be surprises, and there remain intriguing open questions". Our "surprise" is related to the notion of (infinite dimensional) convex polytope, as introduced by Klee ([7]): Proposition 3.2 proves that the convex hull of the extreme points of the unit ball in ℓ_∞ is a polytope, which, by virtue of Proposition 3.6, does not contain any infinite dimensional closed and convex set.

This unusual polytope is an example of a convex F_σ set which cannot be written as a countable union of closed and convex sets, and settles in this way Theorem B when $X = \ell_\infty$; on this ground, Theorem 4.3 addresses the case of a general real Banach space X by using a standard compact embedding result.

The last section of this work addresses a question which, at the best of our knowledge, is open: characterize all the real normed spaces containing a convex F_σ set which cannot be written as a countable union of closed and convex sets.

2. Theorem A

Let us consider X a real Banach space, and $\{D_n : \ n \in \mathbb{N}\}$ a non-decreasing (meaning that $D_n \subseteq D_{n+1}$ for every $n \in \mathbb{N}$) sequence of non empty, closed, convex and bounded subsets of X fulfilling the property (1). The first part of this section is devoted to the proof of the following result.

Proposition 2.1. *Let m be a positive integer, and λ_i, $i = \overline{1,m}$, be non-negative scalars whose sum amounts to 1. Then*

$$\sum_{i=1}^{m} \lambda_i \, D_i \subset D_p, \tag{2}$$

provided that

$$p \geq \sum_{i=1}^{m} i \, \lambda_i. \tag{3}$$

This results plays a special role in achieving Theorem A, as the index p on the right side of the relation (2) depends only upon the coefficients λ_i, $i=\overline{1,m}$, and not upon the sets from the sequence (D_n).

2.1. Proof of Proposition 2.1.

As a first step in establishing the desired result, let us address the following property of non-decreasing sequences of real numbers.

Lemma 2.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of reals such that*

$$x_n + x_{n+2} \le 2\,x_{n+1} \qquad \forall n \in \mathbb{N}. \tag{4}$$

Let m be a positive integer, and λ_i, $i=\overline{1,m}$, be non-negative scalars whose sum amounts to 1. Then

$$\sum_{i=1}^{m} \lambda_i\, x_i \le x_p, \tag{5}$$

provided that relation (3) holds true.

Proof of Lemma 2.2. As the case $m=1$ is clear, we assume that $m \ge 2$. The proof will be done in three steps.

Step 1: We claim that

$$x_n + x_{n+k+1} \le x_{n+1} + x_{x+k} \qquad \forall n, k \in \mathbb{N}. \tag{6}$$

Let us prove relation (6) by induction after k. The case $k=1$ is nothing but property (4), so we may assume that $k>1$ and that property (6) holds for $k-1$:

$$x_n + x_{n+k} \le x_{n+1} + x_{n+k-1} \qquad \forall n \in \mathbb{N}. \tag{7}$$

By combining relation (4), and relation (7) written for $n+1$, we get that

$$\begin{aligned} x_n + x_{x+2} + x_{n+k+1} &\le 2x_{n+1} + x_{n+k+1} \\ &= x_{n+1} + x_{n+1} + x_{n+k+1} \\ &\le x_{n+1} + x_{n+2} + x_{n+k}. \end{aligned} \tag{8}$$

The desired relation (6) obviously follows from relation (8).

Step 2: Let us consider $q, k \in \mathbb{N}$, and $\alpha_i \ge 0$, $i=\overline{q, q+k+1}$. We claim that there exists either $\gamma_i \ge 0$, $i=\overline{q+1, q+k+1}$, with the property that

$$\sum_{i=q+1}^{q+k+1} \gamma_i = \sum_{i=q}^{q+k+1} \alpha_i, \quad \sum_{i=q+1}^{q+k+1} i\,\gamma_i = \sum_{i=q}^{q+k+1} i\,\alpha_i, \tag{9}$$

such that

$$\sum_{i=q}^{q+k+1} \alpha_i\, x_i \le \sum_{i=q+1}^{q+k+1} \gamma_i\, x_i, \tag{10}$$

or $\mu_i \ge 0$, $i = \overline{q, q+k}$, satisfying the relation

$$\sum_{i=q}^{q+k} \mu_i = \sum_{i=q}^{q+k+1} \alpha_i, \quad \sum_{i=q}^{q+k} i\,\mu_i = \sum_{i=q}^{q+k+1} i\,\alpha_i, \tag{11}$$

such that

$$\sum_{i=q}^{q+k+1} \alpha_i\, x_i \le \sum_{i=q}^{q+k} \mu_i\, x_i. \tag{12}$$

Indeed, if $\alpha_q \le \alpha_{q+k+1}$, then

$$\sum_{i=q}^{q+k+1} \alpha_i\, x_i = \alpha_q(x_q + x_{q+k+1}) + \sum_{i=q+1}^{q+k} \alpha_i\, x_i + (\alpha_{q+k+1} - \alpha_q)x_{q+k+1},$$

and, as from the first step of Lemma 2.2 it follows that

$$\alpha_q(x_q + x_{q+k+1}) \le \alpha_q(x_{q+1} + x_{q+k}),$$

we deduce that

$$\begin{aligned} \sum_{i=q}^{q+k+1} \alpha_i\, x_i \quad &\le \quad \alpha_q(x_{q+1} + x_{q+k}) \\ &\quad + \sum_{i=q+1}^{q+k} \alpha_i\, x_i + (\alpha_{q+k+1} - \alpha_q)x_{q+k+1}. \end{aligned} \tag{13}$$

It is easy to see that both the sum of the coefficients, and the sum of the products between the coefficients and their indices is the same on the right and on the left sides of formula (13):

$$\begin{aligned} \alpha_q + \alpha_q + \sum_{i=q+1}^{q+k} \alpha_i + (\alpha_{q+k+1} - \alpha_q) \quad &= \quad \alpha_q + \sum_{i=q}^{q+k+1} \alpha_i - \alpha_q \\ &= \quad \sum_{i=q}^{q+k+1} \alpha_i, \end{aligned} \tag{14}$$

and

$$((q+1)+(q+k))\alpha_q + \sum_{i=q+1}^{q+k} i\,\alpha_i + (q+k+1)(\alpha_{q+k+1}-\alpha_q) = \tag{15}$$
$$(1+(q+k))\alpha_q + \sum_{i=q}^{q+k+1} i\,\alpha_i - (q+k+1)\alpha_q = \sum_{i=q}^{q+k+1} i\,\alpha_i.$$

We have thus established that, in the case $\alpha_q \le \alpha_{q+k+1}$, the coefficients of the right side of relation (13) provide us with a system γ_i, $i=\overline{q+1,q+k+1}$ of non-negative scalars which fulfills the properties (9) and (10). Obviously, when $\alpha_q \ge \alpha_{q+k+1}$, a similar line of reasoning proves that there are $\mu_i \ge 0$, $i=\overline{q,q+k}$, satisfying relations (11) and (12).

Step 3: Let us finally consider the coefficients $\lambda_i \ge 0$, $i=\overline{1,m}$, whose sum equals to 1, and an index p fulfilling relation (3). In order to prove that relation (5) holds true, let us repeatedly apply the procedure given by *Step 2*, with the the coefficients λ_i, $i=\overline{1,m}$, as a starting point. It yields that there is a positive integer $s=\overline{1,m-1}$, and ζ_s, ζ_{s+1}, two non-negative scalars, such that

$$\zeta_s+\zeta_{s+1}=\sum_{i=1}^{m}\lambda_i, \quad s\zeta_s+(s+1)\zeta_{s+1}=\sum_{i=1}^{m} i\,\lambda_i, \tag{16}$$

and

$$\sum_{i=1}^{m}\lambda_i\,x_i \le \zeta_s\,x_s+\zeta_{s+1}\,x_{s+1}.$$

From relations (3) and (14) it results that

$$p \ge \sum_{i=1}^{m} i\,\lambda_i = s\zeta_s+(s+1)\zeta_{s+1}. \tag{17}$$

Let us consider two cases: $\zeta_{s+1}=0$, and $\zeta_{s+1}>0$. When $\zeta_s=0$, then $\zeta_s=1$, so relation (17) becomes $p \ge s$. As the sequence (x_n) is non-decreasing, it follows that $x_s \le x_p$, whence

$$\sum_{i=1}^{m}\lambda_i\,x_i \le \zeta_s\,x_s+\zeta_{s+1}\,x_{s+1} = x_s \le x_p. \tag{18}$$

In the case $\zeta_{s+1} > 0$, then $s\zeta_s + (s+1)\zeta_{s+1} > s$, so from relation (17) it yields that

$$p \geq \sum_{i=1}^{m} i\,\lambda_i = s\zeta_s + (s+1)\zeta_{s+1} > s,$$

that is $p \geq s+1$. Then $x_s \leq x_{s+1} \leq x_p$, so

$$\sum_{i=1}^{m} \lambda_i\, x_i \leq \zeta_s\, x_s + \zeta_{s+1}\, x_{s+1} \leq \zeta_s\, x_p + \zeta_{s+1}\, x_p = x_p. \tag{19}$$

The desired relation 5 may be deduced from statements (18) and (19). $\square$

We are now in a position to address the main technical result of this section.

Proof of Proposition 2.1. Let us pick a continuous linear function $f : X \to \mathbb{R}$ (in other words f is an element of X^*, the topological dual of X), and let us set $x_n := \sup_{D_n} f$, $n \in \mathbb{N}$, (obviously x_n is a real number, since D_n is a bounded set, and f is linear and continuous). The sequence (x_n) is non-decreasing, as the same holds for the sequence (D_n). Moreover, as f is linear, it holds that

$$x_n + x_{n+2} = \sup_{D_n} f + \sup_{D_{n+2}} f = \sup_{D_n + D_{n+2}} f, \tag{20}$$

and that

$$2\,x_{n+1} = 2 \sup_{D_{n+1}} f = \sup_{2\,D_{n+1}} f; \tag{21}$$

but $D_n + D_{n+2} \subseteq 2\,D_{n+1}$, so

$$\sup_{D_n + D_{n+2}} f \leq \sup_{2\,D_{n+1}} f. \tag{22}$$

From relations (20-22) it results that the sequence (x_n) satisfies the property (4), so, by virtue of Lemma 2.2, we deduce that

$$\sum_{i=1}^{m} \lambda_i \sup_{D_i} f \leq \sup_{D_p} f. \tag{23}$$

Again as a consequence of the fact that f is linear, we observe that

$$\sum_{i=1}^{m} \lambda_i \sup_{D_i} f = \sup_{\sum_{i=1}^{m} \lambda_i\, D_i} f; \tag{24}$$

from relations (23) and (24) it yields that

$$\sup_{\sum_{i=1}^{m} \lambda_i D_i} f \leq \sup_{D_p} f \qquad \forall f \in X^*. \tag{25}$$

Being closed and convex, the set D_p is the intersection of all the closed half-spaces containing it; relation (2) follows thus from relation (25). $\square$

2.2. The proof of Theorem A

This subsection characterizes the image of a closed convex set under a Fredholm operator.

Theorem 2.3. *Let X and Y be real Banach spaces, and $\Pi : Y \to X$ be an onto Fredholm operator of positive index. The following statements are equivalent:*
i) the set C is the union of the non-decreasing family of non empty closed and convex sets fulfilling the property (1),
ii) the set C is the image of a non empty closed convex subset of Y under Π.

Proof of Theorem 2.3. $ii) \Rightarrow i)$ Let us consider K, a non empty closed convex subset of Y. Pick $x_0 \in K$, and set $K_n := K \cap (x_0 + n\, B_Y)$, where B_Y is the closed unit ball of Y. It is an easy task to verify that K_n, $n \in \mathbb{N}$, is a non-decreasing sequence of non empty bounded closed and convex subsets of Y fulfilling relation (1). As the image under a Fredholm operator of a bounded closed convex set is always a bounded closed convex set, and as the image under a linear mapping of a non-decreasing sequence satisfying relation (1) is again a non-decreasing sequence satisfying relation (1), it follows that $\Pi(K)$ is the union of an non-decreasing family of non empty closed and convex sets fulfilling the property (1).

$i) \Rightarrow ii)$ We consider D_n, $n \in \mathbb{N}$, a non-decreasing sequence of non empty bounded convex and closed subsets of X which satisfies relation (1). Let V be the kernel of the Fredholm operator Π. As any finite dimensional subspace of a Banach space is complemented, there exists W, a closed subspace of Y disjoint from V, such that $W + V = Y$.

The restriction on W of Π, $\Pi_W : W \to X$, is a bijective continuous linear operator, so the sets $E_n := (\Pi_W)^{-1}(D_n)$ form a non-decreasing sequence of non empty bounded closed and convex subsets of W, and thus of Y; moreover, the elements of this sequence satisfy formula (1).

Let us pick a non null vector $v \in V$ (such a vector always exists since the index of Π, and thus the dimension of V, is a positive integer). We define the sets

$$A := \cup_{i \in \mathbb{N}} (E_i + i\, v)\,,$$

and

$$B := \cup_{i \in \mathbb{N}} (E_i + \{s\, v : \ s \leq i\})\,,$$

and claim that

$$A \subseteq \overline{\mathrm{co}}\,(A) \subseteq B. \tag{26}$$

The sets $H_p := W + \{s\, v : \ s < p\}$, $p \in \mathbb{N}$, are open half-spaces of $W + \mathbb{R}\, v$, which is itself a closed subspace of Y. As A is a subset of $W + \mathbb{R}\, v$, and as H_p is open in $W + \mathbb{R}\, v$, it results that

$$\overline{\mathrm{co}(A)} \cap H_p \subseteq \overline{\mathrm{co}(A) \cap H_p}. \tag{27}$$

Let us pick $x \in \mathrm{co}(A) \cap H_p$. As $x \in \mathrm{co}(A)$, there are some non-negative scalars λ_i, $i = \overline{1, m}$, of sum equal to 1, and some elements $x_i \in E_i$, $i = \overline{1, m}$ such that

$$x = \left(\sum_{i=1}^{m} \lambda_i\, x_i \right) + \left(\sum_{i=1}^{m} i\, \lambda_i \right) w.$$

But $x \in H_p$, so $\sum_{i=1}^{m} i\, \lambda_i < p$; we may thus apply Proposition 2.1 to the sequence (E_n), and deduce that $\sum_{i=1}^{m} \lambda_i\, x_i \subset E_p$. In other words,

$$\mathrm{co}(A) \cap H_p \subseteq E_p + \{s\, v : \ s < p\}. \tag{28}$$

Finally, by combining relations (27) and (28) with the obvious fact that the closure of $E_p + \{s\, v : \ s < p\}$ amounts to $E_p + \{s\, v : \ s \leq p\}$, it yields that

$$\overline{\mathrm{co}(A)} \cap H_p \subseteq E_p + \{s\, v : \ s \leq p\} \qquad \forall p \in \mathbb{N},$$

fact which clearly implies relation (26). Taking into account that

$$\Pi(A) = \Pi(B) = \cup_{i \in \mathbb{N}} \Pi(E_i) = \cup_{i \in \mathbb{N}} D_i,$$

the proof of implication $i) \Rightarrow ii)$ is complete. □

3. The convex hull of the extreme points of the unit ball of ℓ_∞: Theorem B in the case $X = \ell_\infty$

The aim of this section is to prove that the convex hull of the set gathering all the extreme points of the unit ball from the Banach space ℓ_∞ is a convex F_σ set which cannot be expressed as the union of countably many closed and convex sets, proving in this way Theorem B in the case when $X = \ell_\infty$.

To avoid some complicated notation, we will write a typical element of ℓ_∞ as $x := (x(n))_{n=1}^\infty$, and we will call x_n the n-th coefficient of x. Given $S \subset \mathbb{N}$, let us denote be y_S the element of ℓ_∞ defined by the formula

$$y_S(n) := \begin{cases} 1 & n \in S \\ -1 & n \notin S \end{cases}.$$

Obviously, every point of form y_S is an extreme point of B_{ℓ_∞}, the unit ball of ℓ_∞, and every extreme point of B_{ℓ_∞} is of the form y_S:

$$\operatorname{ext} B_{\ell_\infty} = \{y_S : \ S \subseteq \mathbb{N}\}.$$

3.1. Basic properties of $\operatorname{co}(\operatorname{ext} B_{\ell_\infty})$

Let us denote by

$$P(x) := \{x(n) : \ n \in \mathbb{N}\}, \qquad x \in \ell_\infty,$$

the set containing all the coefficients of some vector x from ℓ_∞. For every $S \subset \mathbb{N}$, it is clear that $P(y_S)$ is a subset of $\{-1, \ 1\}$. More generally, the following easy to prove statement provides a characterization of elements x from the convex hull of $\operatorname{ext} B_{\ell_\infty}$ in terms of $P(x)$.

Lemma 3.1. *A vector $x \in X$ belongs to the convex hull of* $\operatorname{ext} B_{\ell_\infty}$ *if and only if $P(x)$ is a finite subset of the closed interval* $[-1, \ 1]$.

Proof of Lemma 3.1. As the *"only if"* part is obvious, let us address the *"if"* part, and consider $x \in X$ such that $P(x)$ is a finite subset of $[-1, \ 1]$. Let α_k, $k = \overline{0, p}$, be the elements of the set $P(x) \cup \{-1, \ 1\}$ written in increasing order; in particular, $\alpha_0 = -1$, and $\alpha_p = 1$. For every $k = \overline{1, p}$ we define

$$\lambda_k := \frac{\alpha_k - \alpha_{k-1}}{2} \quad \text{and} \quad S_k := \{n \in \mathbb{N} : \ x(n) \geq \alpha_k\}. \tag{29}$$

Obviously, all the numbers λ_k, $k = \overline{1,p}$ are positive, and their sum amounts to $\frac{\alpha_p - \alpha_0}{2} = 1$. We claim that

$$x = \sum_{k=1}^{p} \lambda_k \, y_{S_k}. \tag{30}$$

In order to prove our claim, let us pick $n \in \mathbb{N}$; then $x(n) = \alpha_s$ for some $s = \overline{0,p}$. We shall distinguish two cases: $s = 0$ and $s \geq 1$.

Case $s = 0$: Since $x(n) = \alpha_0 = -1$, it results that n does not belong to any of the sets S_k, $k = \overline{1,p}$. Accordingly,

$$\{k = \overline{1,p} : \ y_{S_k}(n) = 1\} = \emptyset,$$

so

$$\sum_{\{k=\overline{1,p}\colon\, y_{S_k}(n)=1\}} \lambda_k = 0 = \frac{\alpha_0 - \alpha_0}{2}. \tag{31}$$

Case $s \geq 1$: With $k = \overline{1,p}$, the following line of equivalent relations is easily seen:

$$k = \overline{1,s} \Leftrightarrow \alpha_k \leq \alpha_s \Leftrightarrow \alpha_k \leq x(n) \Leftrightarrow n \in S_k \Leftrightarrow y_{S_k}(n) = 1. \tag{32}$$

Relation (32) proves that $\{k = \overline{1,p} : \ y_{S_k}(n) = 1\} = \{1, \ldots, s\}$, hence

$$\sum_{\{k=\overline{1,p}\colon\, y_{S_k}(n)=1\}} \lambda_k = \sum_{k=1}^{s} \lambda_k = \frac{\alpha_s - \alpha_0}{2}. \tag{33}$$

By putting together relations (31) and (33), we infer that

$$[x(n) = \alpha_s] \Rightarrow \left[\sum_{\{k=\overline{1,p}\colon\, y_{S_k}(n)=1\}} \lambda_k = \frac{\alpha_s - \alpha_0}{2} \right],$$

fact which allows us to compute the value of $\left(\sum_{k=1}^{p} \lambda_k \, y_{S_k}\right)(n)$:

$$\begin{aligned}
\left(\sum_{k=1}^{p} \lambda_k \, y_{S_k}\right)(n) &= \sum_{k=1}^{p} \lambda_k \, \left(y_{S_k}(n)\right) \\
&= \sum_{\{k=\overline{1,p}:\ y_{S_k}(n)=1\}} \lambda_k - \sum_{\{k=\overline{1,p}:\ y_{S_k}(n)=-1\}} \lambda_k \\
&= 2\left(\sum_{\{k=\overline{1,p}:\ y_{S_k}(n)=1\}} \lambda_k\right) - 1 = 2\,\frac{\alpha_s - \alpha_0}{2} - 1 \\
&= \alpha_s = x(n) \qquad\qquad \forall n \in \mathbb{N}.
\end{aligned}$$

We have thus proved our claim, together with the *"if"* part of Lemma 3.1. $\square$

We are now in a position to prove that $\operatorname{co}(\operatorname{ext} B_{\ell_\infty})$ is a F_σ polytope in the sense of Klee: each and every one of its affine finite-dimensional sections is the convex hull of a finite set of points.

Proposition 3.2. *The following two statements hold true.*
i) the convex hull of the extreme points of the unit ball of ℓ_∞ is a F_σ set,
ii) $\operatorname{co}(\operatorname{ext} B_{\ell_\infty}) \cap L$ *is the convex hull of a finite set, provided that $L \subset \ell_\infty$ is a linear manifold of finite dimension.*

Proof of Proposition 3.2. *i*) Pick $p \in \mathbb{N}$, and let T_p be the set of all the vectors x from ℓ_∞ for which the set $P(x)$ is a subset of $[-1,\ 1]$ containing at most p elements.

Let us also consider a vector y such that $y \notin S_p$. There are two possibilities: $|y(n)| > 1$ for some $n \in \mathbb{N}$, or $P(y) \cap [-1,\ 1]$ contains at least $p+1$ different scalars, say $\{\mu_k :\ k = \overline{1, p+1}\}$.

When it holds that $|y(n)| > 1$, let us set

$$\varepsilon := \frac{|y(n)| - 1}{2},$$

and remark that $|z(n)| > 1$ for every $z \in y + \varepsilon\, B_{\ell_\infty}$, so

$$(y + \varepsilon\, B_{\ell_\infty}) \cap T_p = \emptyset.$$

When $P(y) \cap [-1,\ 1]$ contains the scalars $\{\mu_k :\ k = \overline{1, k+1}\}$, we set

$$\varepsilon := \min \left\{ \frac{|\mu_i - \mu_j|}{3} :\ i \neq j,\ \ i, j = \overline{1, p+1} \right\}.$$

Clearly, the set $P(z) \cap [-1,\ 1]$ contains at least $p+1$ elements, for every $z \in y + \varepsilon\, B_{\ell_\infty}$, so again

$$(y + \varepsilon\, B_{\ell_\infty}) \cap T_p = \emptyset.$$

Consequently, the set T_p is closed for any $p \in \mathbb{N}$; moreover, an obvious consequence of Lemma 3.1 reads that $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) = \cup_{k \in \mathbb{N}} T_k$, so the convex hull of the extreme points of the unit ball of ℓ_∞ is a F_σ set.

ii) Given $L \subset \ell_\infty$, a linear manifold of finite dimension, we denote by p the dimension of the set $C := \mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) \cap L$; then C contains $p+1$ vectors, say x_k, $k = \overline{0, p}$, such that $C = \mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) \cap L'$, where

$$L' := \left\{ x_0 + \sum_{k=1}^{p} \gamma_k\, (x_k - x_0) :\ (\gamma_k)_{k = \overline{1,p}} \in \mathbb{R}^p \right\}$$

is the linear manifold spanned by the set $\{x_k :\ k = \overline{0, p}\}$.

By virtue of Lemma 3.1, it holds that C may be seen as the set of all the elements x from L' such that $P(x)$ is a finite subset of $[-1,\ 1]$. But the set $P(x)$ is finite for any $x \in L'$. Indeed, given $n \in \mathbb{N}$, we define the following vector from $\mathbb{R}^p$:

$$\boldsymbol{w_n} := (x_k(n) - x_0(n))_{k \in \overline{1,p}};$$

as for every $k = \overline{0, p}$, $P(x_k)$ is a finite subset of $[-1,\ 1]$, it easily yields that the set $W := \{\boldsymbol{w_n} :\ \ n \in \mathbb{N}\}$ is a finite subset of the p-dimensional cube $[-2,\ 2]^p$. Given $x \in L'$, there exists $\boldsymbol{\gamma} := (\gamma_k)_{k \in \overline{1,p}} \in \mathbb{R}^p$ such that

$$x = x_0 + \sum_{k=1}^{p} \gamma_k\, (x_k - x_0);$$

thus

$$x(n) = x_0(n) + \sum_{k=1}^{p} \gamma_k\, (x_k(n) - x_0(n)) = x_0(n) + \langle \boldsymbol{\gamma},\ \boldsymbol{w_n} \rangle$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is the standard dot product in $\mathbb{R}^p$. As the vector x_0 has only a finite number of different coefficients, and since $\mathbf{w_n}$, $n \in \mathbb{N}$, runs through the finite set W, it results that the vector x has also a finite number of different coefficients.

Consequently, the set $C = \mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) \cap L$ may be seen as the set of all the elements x from L' such that $P(x) \subset [-1,\ 1]$:

$$C = \left\{ x_0 + \sum_{k=1}^{p} \gamma_k \,(x_k - x_0) :\ \gamma \in M \right\},$$

where

$$M := \{\gamma \in \mathbb{R}^p :\ -1 \le x_0(n) + \langle \boldsymbol{\gamma},\ \boldsymbol{w_n} \rangle \le 1 \quad \forall n \in \mathbb{N}\}.$$

As already observed, both the sets W and $P(x_0)$ are finite, so the set M is the intersection of finitely many closed half-spaces of $\mathbb{R}^p$. Hence C, a bounded subset of L', is the intersection of finitely many closed half-spaces of L', and an application of Weyl-Minkowsky's theorem concludes the proof of point ii). $\square$

Remark 3.3. It worth noticing that the polytope $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$ has two unusual properties. First, when the finite dimensional linear manifold L runs through the class of all the linear manifolds of a given dimension, the number of vertices of the bounded convex polytope $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) \cap L$ is not bounded from above. More precisely, given $n \in \mathbb{N}$, it is easy to find a two-dimensional linear manifold L such that the convex polygon $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) \cap L$ has at least n vertices.

Secondly, its closure, that is the unit ball in ℓ_∞, is not a polytope.

3.2. Infinite dimensional subsets of $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$

A key step in achieving the main result of this section is to characterize the infinite dimensional subsets of the convex hull of $\mathrm{ext}\, B_{\ell_\infty}$. To this purpose, we denote by

$$R(\mu, x) := \{n \in \mathbb{N} :\ x(n) = \mu\} \qquad x \in \ell_\infty,\ \mu \in\ P(x),$$

the set of all indices $n \in \mathbb{N}$ at which the n-th coefficient of x takes the value μ, and, given $A \subset \ell_\infty$, we set

$$R(A) := \{R(\mu, x) :\ x \in A,\ \mu \in\ P(x)\}.$$

Proposition 3.4. *Let A be a subset of the convex hull of $\operatorname{ext} B_{\ell_\infty}$. The following sentences are equivalent:*

i) the dimension of A is infinite,

ii) the set $R(A)$ is infinite,

iii) it exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$, and, for any $n \in \mathbb{N}$, a scalar $\mu_n \in P(x_n)$, such that the sets

$$K_n := \cap_{i=1}^{n} R(\mu_i, x_i) \tag{34}$$

form a decreasing sequence.

Proof of Proposition 3.4. As obviously *iii*) $\Rightarrow$ *ii*), all what we have to show is that *i*) $\Rightarrow$ *ii*), *ii*) $\Rightarrow$ *i*), and that *ii*) $\Rightarrow$ *iii*).

i) $\Rightarrow$ *ii*): Let us consider $A \subset \operatorname{co}(\operatorname{ext} B_{\ell_\infty})$ such that $R(A)$ is a finite set. Thus, $\mathfrak{R}$, the ring of sets spanned by $\{\emptyset\} \cup R(A)$ (we use here the notion of ring of sets in Birkhoff's sense, meaning a family of sets closed under the operations of set unions and set intersections) is itself finite. We set $Y_A := \{y_S : S \in \mathfrak{R}\}$, and we claim that $A \subset \operatorname{co}(Y_A)$.

Indeed, let us pick $x \in A$; as already seen (relations (29) and (30)), x lies within the convex hull of the set $\{y_{S_k} : k = \overline{1,p}\}$, where $p+1$ is the number of elements from the set $P(x) \cup \{-1,\ 1\}$, the scalars α_k, $k = \overline{0,p}$, are the elements of $P(x) \cup \{-1,\ 1\}$ written in increasing order, and $S_k = \{n \in \mathbb{N} : x(n) \geq \alpha_k\}$.

It remains to prove that the sets S_k, $k = \overline{1,p}$ are elements of the ring $\mathfrak{R}$. Of course, $\alpha_k \in P(x)$ for every $k = \overline{1,p-1}$, so

$$S_k = \left(\cup_{i=k}^{p-1} R(\alpha_k, x)\right) \cup S_p, \qquad \forall k = \overline{1,p-1}.$$

As moreover

$$S_p = \begin{cases} \emptyset & 1 \notin P(x) \\ R(1,x) & 1 \in P(x) \end{cases} \qquad \forall p \in \mathbb{N},$$

we may conclude that, for any $k = \overline{1,p}$, the set S_k is either the empty set, or the union of finitely many sets of form $R(\mu, x)$, with $\mu \in P(x)$; in both cases, $S_k \in \mathfrak{R}$.

Accordingly, $\{y_{S_k} : k = \overline{1,p}\} \subset Y_A$, fact which proves our claim, and consequently the fact that the set A is finite, provided that $R(A)$ is a finite set. The *i*) $\Rightarrow$ *ii*) part of Proposition 3.4 is established.

ii) $\Rightarrow$ *i*): Let us pick A, a finite dimensional subset of $\operatorname{co}(\operatorname{ext} B_{\ell_\infty})$. By virtue of Proposition 3.2, it follows that A is contained in the convex hull of a finite subset of $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$: in other words, there exists a positive integer $p \in \mathbb{N}$, and $\{x_k :\ k = \overline{1,p}\} \subset \mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$ such that $A \subset \mathrm{co}\,\{x_k :\ k = \overline{1,p}\}$.

Clearly, both the set $R\left(\{x_k :\ k = \overline{1,p}\}\right)$, and $\mathfrak{R}$, the ring of sets spanned by $R\left(\{x_k :\ k = \overline{1,p}\}\right)$, are finite. Let us pick

$$\boldsymbol{\mu} := (\mu_k)_{k\in\overline{1,p}} \in \Pi_{k=1}^{p} P(x_k);$$

the set

$$B_{\boldsymbol{\mu}} := \cap_{k=1}^{p} R(\mu_k, x_k),$$

besides providing an obvious example of an element of $\mathfrak{R}$, has the following property:

$$x_k(n) = x_k(m) \quad \forall k = \overline{1,p},\ \forall n, m \in B_{\boldsymbol{\mu}}. \tag{35}$$

As every element x from A may be expressed as a convex combination of the vectors x_k, $k = \overline{1,p}$, from relation (35) it yields that

$$x(n) = x(m) \quad \forall x \in\ A,\ \forall\ n, m \in\ B_{\boldsymbol{\mu}};$$

consequently, every set of form $R(\mu, x)$ with $x \in A$ and $\mu \in P(x)$ is necessarily the union of a family of sets of form $B_{\boldsymbol{\mu}}$, with $\boldsymbol{\mu} \in \Pi_{k=1}^{p} P(x_k)$.

We have thus proved that

$$R(\mu, x) \in \mathfrak{R} \qquad \forall x \in A,\ \forall \mu \in P(x),$$

relation which entails that $R(A)$ is a subset of $\mathfrak{R}$, hence a finite set, for every finite dimensional set A.

ii) $\Rightarrow$ *iii)*: To the purpose of establishing this implication, we will repeatedly use the following result, whose obvious proof will be omitted.

Lemma 3.5. *Let J be an infinite subset of $\mathbb{N}$, and $L \subset 2^J$ be an infinite set of subsets of J. Given $J = \cup_{i=1}^{p} J_i$ a finite partition of J, there is at least one index $k = \overline{1,p}$ such that the set $\{D \cap J_k :\ D \in L\}$ is an infinite set of subsets of J_k.*

Let us consider A, a subset of $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$ such that the set $R(A)$ is infinite. We will inductively define a sequence $(x_n)_{n\in\mathbb{N}}$ of vectors from A, and, for every $n \in \mathbb{N}$, a scalar $\mu_n \in P(x_n)$, such that the two following relations hold true:
α): the sets K_i, $i = \overline{1,n}$, given by formula (34), are all different,
β) the set $\{D \cap K_n :\ D \in R(A)\}$ is infinite.

Case $n = 1$: Let us pick $x_1 \in A$. As relation α) obviously holds true when $n = 1$, we only need to define $\mu_1 \in P(x_1)$ such that the set $\{D \cap K_1 : D \in R(A)\}$ is finite.

To this respect, we apply Lemma 3.5 for $J := \mathbb{N}$, $L := R(A)$, and the finite partition

$$\mathbb{N} = \cup_{\mu \in P(x_1)} R(\mu, x_1)$$

of $\mathbb{N}$, and we deduce that it is always possible to pick $\mu_1 \in P(x_1)$ such that the set

$$\{D \cap R(\mu_1, x_1) : \ D \in R(A)\} = \{D \cap K_1 : \ D \in R(A)\}$$

is infinite.

Case $n > 1$: Having defined x_i and μ_i, $i = \overline{1, n-1}$, let us remark that all what it is needed in order to fulfill relation α), is to pick the vector $x_n \in A$ in such a way that the set $\{K_{n-1} \cap R(\mu, x_n) : \ \mu \in P(x_n)\}$ possesses at least two elements. Or, it is always possible to pick such a vector x_n from A; indeed, by assuming, to the end of achieving a contradiction, that for every $x \in A$ it holds that $K_{n-1} \subset R(\mu, x)$ for some $\mu \in P(x)$, then it results that

$$\{D \cap K_{n-1} : \ D \in R(A)\} = \{K_{n-1}\}.$$

But relation β) is valid for $n-1$, so the set $\{D \cap K_{n-1} : \ D \in R(A)\}$ is infinite, and cannot contain only one element; our assumption is accordingly false.

It remains to chose the scalar $\mu_n \in P(x_n)$ such that relation β) is satisfied; to this respect, let us apply once again Lemma 3.5, this time for $J := K_{n-1}$, $L := \{D \cap K_{n-1} : \ D \in R(A)\}$ and for the finite partition

$$K_{n-1} = \cup_{\mu \in P(x_n)} \left(K_{n-1} \cap R(\mu, x_n)\right).$$

It follows that there is an element, say μ_n, in $P(x_n)$ such that the set

$$\{D \cap (K_{n-1} \cap R(\mu_n, x_n)) : \ D \in L\} = \{D \cap K_n : \ D \in R(A)\}$$

is infinite, and the second case is completed. Sequences (x_n) and (μ_n) obviously prove implication $ii) \Rightarrow iii)$. $\square$

On the ground of Proposition 3.4, we establish the following result, providing a key property of the convex hull of $\mathrm{ext}(B_{\ell_\infty})$.

Proposition 3.6. *The convex hull of the extreme points of the unit ball of ℓ_∞ does not contain any infinite dimensional closed convex set.*

Proof of Proposition 3.6. Let us consider A, an infinite dimensional subset of $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$. By virtue of implication $i) \Rightarrow iii)$ from Proposition 3.4, there is (x_n), a sequence of elements from A, and (μ_n), $\mu_n \in P(x_n)$, a sequence of scalars, such that (K_n), the sequence defined by formula (34), is decreasing. Our objective is to define a sequence of positive real numbers λ_n, $n \in \mathbb{N}$, of sum equal to 1, such that the vector $\mathbf{x} := \sum_{n\in\mathbb{N}} \lambda_n\, x_n$ possesses infinite many different coefficients.

As the sequence (K_n) is decreasing, it is always possible to pick, for every $n \in \mathbb{N}$, an index a_n which lies in K_n, but outside K_{n+1}; accordingly,

$$x_n(a_k) = x_n(a_p) \qquad \forall n, k, p \in \mathbb{N} \text{ with } n \le k < p, \tag{36}$$

and

$$x_{n+1}(a_n) \neq x_{n+1}(a_k) \qquad \forall n, k \in \mathbb{N} \text{ with } n < k. \tag{37}$$

It is an easy task to recursively construct the sequence of positive scalars τ_n, $n \in \mathbb{N}$, such that $\tau_1 := 1$, $\tau_2 := 1$, and that

$$\tau_n \le \frac{\tau_{q+1}}{2^n}\, |x_{q+1}(a_q) - x_{q+1}(a_{q+1})|, \quad \forall q, n \in \mathbb{N} \text{ with } q+1 < n. \tag{38}$$

Indeed, let $n \ge 3$ be a positive integer, and assume that the numbers $\tau_i > 0$, $i = \overline{1, n-1}$ have already been chosen. As a consequence of relation (37) we deduce that

$$\frac{\tau_{q+1}}{2^n}\, |x_{q+1}(a_q) - x_{q+1}(a_{q+1})| > 0, \qquad \forall q \in \overline{1, n-2};$$

it is thus always possible to pick a positive scalar τ_n fulfilling all the $n-2$ inequalities implied by relation (38).

All the coefficients of all the elements x_n, $n \in \mathbb{N}$, lay between -1 and 1; in particular, $|x_2(a_1) - x_2(a_2)| \le 2$. By applying relation (38) for $q = 1$, $n \ge 3$, one deduces that

$$\tau_n \le \frac{\tau_2}{2^n}|x_2(a_1) - x_2(a_2)| \le \frac{1}{2^{n-1}} \qquad \forall n \in \mathbb{N},\ n \ge 3,$$

so the series $\left(\sum_{n\in\mathbb{N}} \tau_n\right)$ is convergent. We define $\lambda_n := \frac{\tau_n}{\sum_{n\in\mathbb{N}} \tau_n}$, a sequence of positive real numbers whose sum is equal to 1. The following well-known and easy to prove fact is provided here without a proof.

Lemma 3.7. *Let X be a real Banach space, $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence, and $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence of non-negative real numbers whose sum is equal to 1. Then the series $\left(\sum_{k=1}^{+\infty} \lambda_k\, x_k\right)$ is convergent, and its sum belongs to the closed convex hull of the set $\{x_n :\ n \in \mathbb{N}\}$.*

The series $\left(\sum_{n\in\mathbb{N}} \lambda_n\, x_n\right)$ is thus convergent; let $\mathbf{x}$ denote its sum. We claim that the coefficients $\mathbf{x}(a_n)$, $n \in \mathbb{N}$, are all different.

To prove our claim, let us consider two positive integers q and k such that $q < k$. By denoting the sum of the series $\left(\sum_{i\in\mathbb{N}} \tau_i\right)$ by s, we have

$$\begin{aligned} s\,\mathbf{x}(a_q) &= \sum_{n\in\mathbb{N}} \tau_i\, x_i(a_q) \\ &= \sum_{n=1}^{q} \tau_i\, x_i(a_q) + \tau_{q+1}\, x_{q+1}(a_q) + \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_q), \end{aligned} \tag{39}$$

and

$$\begin{aligned} s\,\mathbf{x}(a_k) &= \sum_{n\in\mathbb{N}} \tau_i\, x_i(a_k) \\ &= \sum_{n=1}^{q} \tau_i\, x_i(a_k) + \tau_{q+1}\, x_{q+1}(a_k) + \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_k). \end{aligned} \tag{40}$$

From relation (36) it results that

$$\sum_{n=1}^{q} \tau_i\, x_i(a_q) = \sum_{n=1}^{q} \tau_i\, x_i(a_k), \tag{41}$$

and that

$$x_{q+1}(a_k) = x_{q+1}(a_{q+1}). \tag{42}$$

Putting together relations (39-42) we obtain that

$$\begin{aligned} s\ (\mathbf{x}(a_q) - \mathbf{x}(a_k)) = \quad & \tau_{q+1}\ (x_{q+1}(a_q) - x_{q+1}(a_{q+1})) \\ & + \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_q) - \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_k), \end{aligned} \tag{43}$$

so, by taking the absolute value in relation (43) we infer that

$$\begin{aligned} |s\ (\mathbf{x}(a_q) - \mathbf{x}(a_k))| \geq \quad & |\tau_{q+1}\ (x_{q+1}(a_q) - x_{q+1}(a_{q+1}))| \\ & - \left| \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_q) - \sum_{n=q+2}^{\infty} \tau_i\, x_i(a_k) \right|. \end{aligned} \tag{44}$$

Let us estimate the two absolute values from the right side of relation (44). As the coefficients of the vectors x_n are contained between -1 and 1, and since the numbers λ_n are positive, we may conclude that

$$\left| \sum_{n=q+2}^{\infty} \tau_i \, x_i(a_q) - \sum_{n=q+2}^{\infty} \tau_i \, x_i(a_k) \right| \leq 2 \sum_{n=q+2}^{\infty} \tau_i. \tag{45}$$

on the other hand, by summing up after $n \geq q+2$ relation (38), one gets

$$\begin{aligned} \sum_{n=q+2}^{\infty} \tau_i \quad & \leq \quad \left(\sum_{n=q+2}^{\infty} \frac{1}{2^n} \right) \tau_{q+1} \left| x_{q+1}(a_q) - x_{q+1}(a_{q+1}) \right| \\ & = \quad \frac{\tau_{q+1} \left| x_{q+1}(a_q) - x_{q+1}(a_{q+1}) \right|}{2^{q+1}} \\ & < \quad \frac{\left| \tau_{n+1} \ (x_{n+1}(a_n) - x_{n+1}(a_{n+1})) \right|}{2}. \end{aligned} \tag{46}$$

Taking into account relations (45) and (46), it yields that

$$\begin{aligned} & \left| \sum_{n=q+2}^{\infty} \tau_i \, x_i(a_n) - \sum_{n=q+2}^{\infty} \tau_i \, x_i(a_k) \right| \\ & < \left| \tau_{q+1} \ (x_{q+1}(a_q) - x_{q+1}(a_{q+1})) \right|; \end{aligned} \tag{47}$$

finally, relations (44) and (47) combine to prove our claim.

The vector $\mathbf{x}$ has accordingly an infinite number of different coefficients. Lemma 3.1 proves that $\mathbf{x}$ does not belong to the convex hull of the extreme points of the unit ball of ℓ_∞, while Lemma 3.7 shows that $\mathbf{x}$ is an element of the closed convex hull of the set A: consequently, the convex hull of the extreme points of B_{ℓ_∞} does not contain any infinite dimensional closed convex set. $\square$

Remark 3.8. A different light may be thrown on the previous result by recalling the well-known theorem by Fonf and Lindenstrauss ([5, Theorem 3.3]) which proves that the norm interior of any symmetric closed polytope is non empty, provided that the polytope does not lay in any closed hyperplane in X.

Or Proposition 3.6 proves that the convex hull of the extreme points of the unit ball in ℓ_∞, a set which, by virtue of Proposition 3.2, is a symmetric polytope, and visibly does not lay in any closed hyperplane in ℓ_∞, behaves in a strikingly different manner. Indeed, not only the norm interior of this set is empty, but $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$ does not contain any infinite dimensional closed and convex set; in particular, the norm interior of every one of its infinite-dimensional sections is empty.

3.3. Theorem B for the case $x = \ell_\infty$

The conclusions of Proposition 3.6 allow us to prove that Theorem B holds true in ℓ_∞.

Theorem 3.9. *The convex hull of the extreme points of the unit ball of ℓ_∞ cannot be expressed as a countable union of closed and convex sets.*

Proof of Theorem 3.9. To the end of achieving a contradiction, let us assume that

$$\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty}) = \cup_{n\in\mathbb{N}}\, C_n,$$

where C_n, $n \in \mathbb{N}$ are closed and convex subsets of ℓ_∞.

The set $\{y_S : \ S \subseteq \mathbb{N}\}$ of all the extreme points of the unit ball of ℓ_∞ is infinite and uncountable; accordingly, there is at least one of the sets C_n, $n \in \mathbb{N}$, say C_{n_0}, which contains infinitely many vectors of form y_S, $S \subseteq \mathbb{N}$.

Consequently, the set $R(C_{n_0})$ is infinite, and implication $ii) \Rightarrow i)$ from Proposition 3.4 proves that the set C_{n_0} is infinite dimensional. We have thus found a closed and convex subset of $\mathrm{co}\,(\mathrm{ext}\, B_{\ell_\infty})$ whose dimension is infinite, fact which contradicts the conclusions of Proposition 3.6, proving in this way that our initial assumption is false. $\square$

4. Theorem B

This section addresses Theorem B in the general setting of real Banach spaces. In order to construct, in any given real Banach space X, a convex F_σ set which cannot be written as the union of countably many closed and convex sets, we use the well-known theorem by Mazur [1] which says that any infinite dimensional Banach space contains a basic sequence.

[1] Originally stated by Banach without any proof at page 238 of his foundational treatise [2], this result was proved in [3, Corollary 3, page 157]; the reader may found a complete account of this topic in the chapter V of the textbook [4]

The proofs of the following two easy exercises will be omitted.

Lemma 4.1. *Let X be an infinite dimensional real Banach space, and x_n, $n \in \mathbb{N}$, be a basic sequence in X. The operator $K : \ell_\infty \to X$ defined by the formula:*

$$K(y) = \sum_{n\in\mathbb{N}} \frac{y(n)}{2^n}\, x_n \qquad \forall y \in \ell_\infty$$

is one-to-one an compact.

Lemma 4.2. *Let Y and X be two real Banach spaces, and $K : Y \to X$ be a one-to-one compact operator. If $A \subset Y$ is a convex F_σ set which cannot be expressed as the union of countably many closed and convex sets, then the same holds true for $K(A) \subset X$.*

Theorem 4.3. *In any infinite dimensional real Banach space, there is a convex F_σ set which cannot be expressed as a countable union of closed and convex sets.*

Proof of Theorem 4.3. The desired conclusion easily follows by combining the conclusions of Lemmata 4.1 and 4.2, and of Theorem 3.9. □

5. Concluding remarks

Theorems A and B provide us with a good understanding of the properties of the image of a closed convex subset of a real Banach space under a Fredholm operator. These results may accordingly be seen as a generalization of the analysis done by Klee in [6, Theorem 6.1] for the Euclidean case.

The study of Theorem B naturally rises the following question: does any infinite dimensional real normed space contains a convex F_σ set which cannot be expressed as the union of countably many closed and convex sets? The answer, as proved by the following result, is negative.

Proposition 5.1. *Let X be a real normed space, and assume that $X = \cup_{n\in\mathbb{N}} X_n$, where X_n are finite dimensional subspaces of X. Then every convex F_σ set may be expressed as the union of countably many closed and convex sets.*

Proof of Proposition 5.1. Let $C \subseteq X$ be a convex set such that $C = \cup_{n\in\mathbb{N}} F_n$, where F_n, $n \in \mathbb{N}$, are closed sets, and let us pick $x_0 \in C$.

Let us set $D_n := (\cup_{i=1}^{n} F_i) \cap (x_0 + X_n) \cap (x_0 + n\, B_X)$; each of the sets D_n is a no empty finite dimensional compact set, so its convex hull is closed. As moreover $C = \cup_{n\in\mathbb{N}} D_n$ and C is convex, it follows that $C = \cup_{n\in\mathbb{N}} \operatorname{co} D_n$. $\square$

We are thus lead to address the following question, which, at the best of our knowledge is open: which are the real normed spaces containing a convex F_σ set which cannot be expressed as the union of countably many closed and convex sets? In view of Theorem B and Proposition 5.1, the answer must include the infinite dimensional Banach spaces, but must exclude the normed spaces which can be written as the union of countably many of their finite dimensional subspaces.

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